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OPTIMUM DIMENSIONS OF UNIFORM ANNULAR FINS

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Abstract—An equation is derived relating the optimum dimensions of uniform annular fins to the heat transfer and thermal properties of the fin and the heat-transfer coefficient between the fin and coolant. Some computed results covering fairly wide ranges of fin properties and coolant heat-transfer coefficient are presented graphically in terms of suitable dimensionless parameters. The choice of parameters makes it possible to determine the optimum dimensions of a fin, i.e. the width, length and volume, in terms of the total heat to be dissipated by the fin, its thermal conductivity and bore radius, the heat-transfer coefficient between the fin and coolant, and the difference in temperature between the fin bore and coolant. The generalized results are applied to a specific example which shows the effect of material properties upon the optimum dimensions. Finally, a comparison is made between the optimum dimensions of uniform fins and the dimensions of fins of least material.

NOMENCLATURE

b, thickness of fin;
h, heat-transfer coefficient between fin and coolant;
k, thermal conductivity of fin material;
q_{tot}, total heat dissipated from the fin;
r, arbitrary radius;
T, temperature of fin at radius r;
T_c, temperature of coolant;
$\theta = T - T_c$;
V, fin volume;
$u = (2hr_0/k)^{\frac{1}{2}}$, a parameter determined by heat-transfer coefficient and conductivity;
$v = V/\pi r_0^3$, a parameter specifying the fin volume;
$w = r_0/b$, a parameter specifying the fin thickness;
$Q = q_{tot}/k2\pi r_0\theta_0$, a parameter specifying the total heat dissipated from the fin;
$\eta = q_{tot}/2\pi(r_1^2 - r_0^2)h\theta_0$, effectiveness of fin;
I_ν, K_ν, νth order modified Bessel functions of first and second kinds;
$Z = (2h/kb)^{\frac{1}{2}} r$;

subscripts 0 and 1 refer to inner and outer radii of the fin.

INTRODUCTION

The optimization of straight fins has been considered in some detail in a number of references [1, 2, 3], whereas annular fins have received much less attention. Annular fins are of considerable practical importance occurring, for example, on fuel cans in nuclear reactors. The only attempts at optimizing such fins have been concerned with determining the profile of the fin of least material which will dissipate a given quantity of heat. It was found that the fin thickness b varied with radius according to the equation

$$\frac{kb}{2hr_1^2} = \frac{1}{3}\left(\frac{r}{r_1}\right)^2 - \frac{1}{2}\left(\frac{r}{r_1}\right) + \frac{1}{6}\left(\frac{r_1}{r}\right)$$

where r_1 is given by

$$q_{tot} = \frac{2}{3}\pi h\theta_0 r_0^2\left(\frac{r_1}{r_0} + 2\right)\left(\frac{r_1}{r_0} - 1\right)$$

This type of optimization is of academic interest but of little use in practice. The manufacture of fins with such complex profiles would be exceedingly difficult and costly.

An annular fin should be considered in the same way as a straight fin, i.e. given a profile, say uniform, triangular, exponential, etc., determine its optimum dimensions. The uniform annular fin is the most commonly used, and in

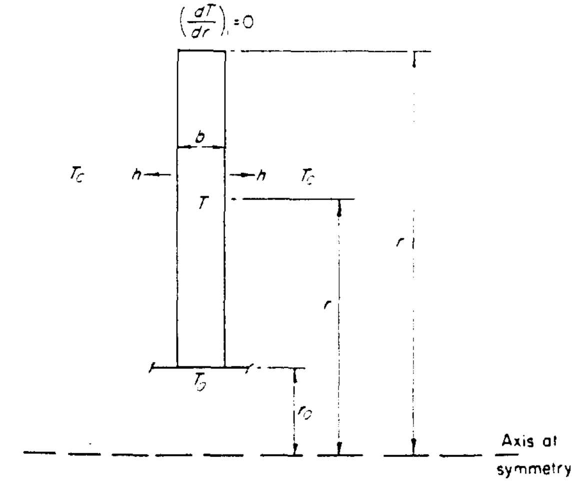

FIG. 1. Diagram showing notation.

this paper its optimum dimensions are determined. The equations relating the fin dimensions to the heat transfer and thermal properties of the fin and the heat-transfer coefficient between the fin and its surroundings are rather complex. However, with the use of Bessel function tables [4], the optimum dimensions have been calculated over a suitable working range and illustrated graphically.

THEORY

Consider the uniform annular fin of Fig. 1, to which heat is supplied at the bore, where its temperature is T_0, and is removed by forced convection of a coolant at temperture T_c. It is assumed that the fin is sufficiently thin for axial temperature variation to be negligible, that the heat-transfer coefficient h between fin and coolant is constant and that there is no radial heat flux at the outer radius, i.e.

$$\left(\frac{\mathrm{d}T}{\mathrm{d}r}\right)_1 = 0 \tag{1}$$

The total heat dissipated from the fin is given by

$$q_{\text{tot}} = k2\pi r_0\theta_0\sqrt{\left(\frac{2hb}{k}\right)} \left(\frac{I_1(Z_0)K_1(Z_1) - K_1(Z_0)I_1(Z_1)}{I_0(Z_0)K_1(Z_1) + K_0(Z_0)I_1(Z_1)}\right) \tag{2}$$

The optimum dimensions of annular fins are those dimensions which give the greatest amount of heat dissipation for a given quantity of material. If V is the volume of material, equation (2) can be expressed as

$$q_{\text{tot}} = -k2\pi r_0\theta_0\sqrt{\left(\frac{2hb}{k}\right)} \left\{\frac{I_1\left(\sqrt{\frac{2hr_0^2}{kb}}\right)K_1\left[\sqrt{\frac{2hr_0^2}{kb}\left(1+\frac{V}{\pi r_0^2 b}\right)}\right] - K_1\left(\sqrt{\frac{2hr_0^2}{kb}}\right)I_1\left[\sqrt{\frac{2hr_0^2}{kb}\left(1+\frac{V}{\pi r_0^2 b}\right)}\right]}{I_0\left(\sqrt{\frac{2hr_0^2}{kb}}\right)K_1\left[\sqrt{\frac{2hr_0^2}{kb}\left(1+\frac{V}{\pi r_0^2 b}\right)}\right] + K_0\left[\sqrt{\frac{2hr_0^2}{kb}}\right]I_1\left[\sqrt{\frac{2hr_0^2}{kb}\left(1+\frac{V}{\pi r_0^2 b}\right)}\right]}\right\} \tag{3}$$

(3) expresses how the heat flow varies for different fin thicknesses. Keeping h, k, θ_0, r_0 and V constant and considering b as the only independent variable, the heat dissipated from the fin is a maximum when $\mathrm{d}q_{\text{tot}}/\mathrm{d}b = 0$. Differentiating equation (3) with respect to b and equating to zero and then re-arranging gives

$$2 + Z_0\left[\frac{I_1(Z_0)\,K_1(Z_1) - K_1(Z_0)\,I_1(Z_1)}{I_0(Z_0)\,K_1(Z_1) + K_0(Z_0)\,I_1(Z_1)} - \frac{I_0(Z_0)\,K_1(Z_1) + K_0(Z_0)\,I_1(Z_1)}{I_1(Z_0)\,K_1(Z_1) - K_1(Z_0)\,I_1(Z_1)}\right]$$

$$+ Z_1\left(1 + \frac{Z_0^2 V}{Z_1^2\pi r_0^2 b}\right)\left\{\frac{[I_0(Z_0)\,K_1(Z_0) + I_1(Z_0)\,K_0(Z_0)]\,[I_0(Z_1)\,K_1(Z_1) + I_1(Z_1)\,K_0(Z_1)]}{[I_0(Z_0)\,K_1(Z_1) + I_1(Z_1)\,K_0(Z_0)]\,[I_1(Z_0)\,K_1(Z_1) - K_1(Z_0)\,I_1(Z_1)]}\right\} = 0 \tag{4}$$

where

$$Z_0 = (2h/kb)^{\frac{1}{2}}\, r_0$$

and

$$Z_1 = (2h(1 + V/\pi r_0^2 b)/kb)^{\frac{1}{2}}\, r_0$$

The optimum dimensions of annular fins are obtained from the solution of equation (3) and (4) and are shown in Fig. 2.

The effectiveness of a fin is the ratio of heat dissipated from the fin to the heat that would be dissipated if the entire fin surface area was maintained at the same temperature as the fin bore. Thus, the effectiveness of uniform annular fins of optimum dimensions, neglecting the heat loss from the tip, is given by

$$\eta = \frac{q_{tot}}{2\pi(r_1^2 - r_0^2)\, h\theta_0} = \frac{2Q}{u^2 v w} \qquad (5)$$

DISCUSSION

The interpretation of Fig. 2 is made clearer by considering the following example. Determine the optimum dimensions of uniform annular fins which will dissipate 500 chu/h when the heat-transfer coefficient between fin and coolant is 50 chu/h ft^2 degC, the bore radius is 1 in, and the difference in temperature between the fin bore and coolant is 200°C, for a number of different fin materials. The physical properties of the various materials and the optimum dimensions of the corresponding fins, as obtained from Fig. 2, are listed in Table 1. It is found that the fin constructed of the material with the highest thermal conductivity, i.e. copper, requires the least volume to dissipate a given amount of heat. However, if weight is the criterion used to measure the required quantity of material then aluminium is the most efficient. In practice, more account is taken of physical and mechanical material

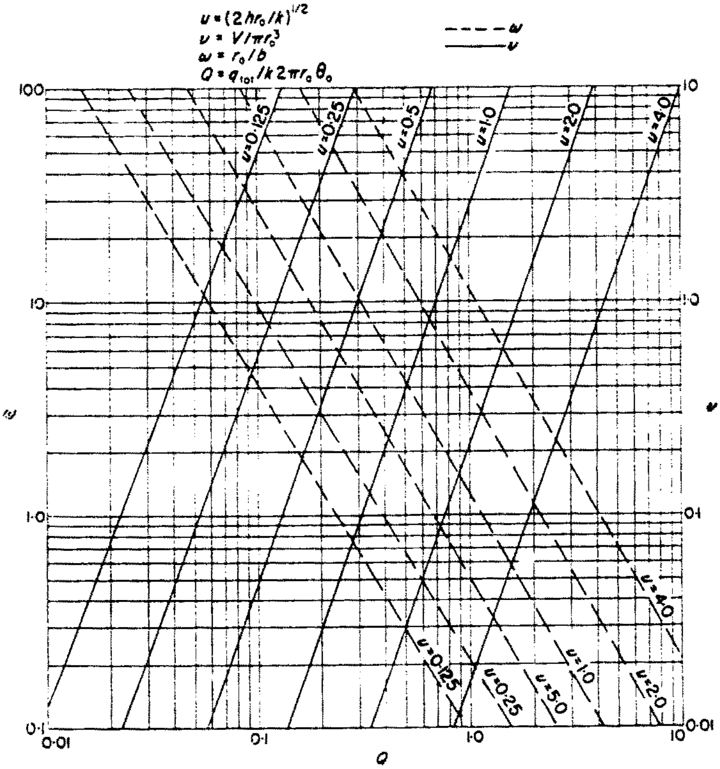

FIG. 2. Optimum dimensions of uniform annular fins.

Table 1. Optimum dimensions of uniform annular fins

Material	Thermal conductivity (chu/h ft degC)	Density (lb/in.^{3})	Q	u	v	w	Volume V (in^3)	Weight of fin / Weight of aluminium fin	Thickness b (in)	Tip radius r_1 (in)	Fin effectiveness
Copper	223	0·0323	0·0214	0·193	0·022	86	0·068	1·9	0·0116	1·69	0·609
Aluminium	132	0·098	0·0362	0·251	0·0375	51	0·118	1·0	0·0196	1·71	0·602
Brass	64	0·308	0·0746	0·360	0·071	27	0·223	5·94	0·0370	1·71	0·602
Mild steel	31	0·283	0·154	0·518	0·146	13	0·459	11·25	0·0770	1·70	0·605
Stainless steel	9·4	0·283	0·508	0·941	0·465	4·05	1·46	35·8	0·247	1·70	0·609
									Arithmetic mean:	1·70	0·605

Table 2. Dimensions of annular fins of least material

Material	Thermal conductivity (chu/h ft degC)	Density (lb/in^3)	Volume V (in^3)	Weight of fin / Weight of aluminium fin	Bore thickness b_0 (in)	Tip radius r_1 (in)	Fin effectiveness
Copper	223	0·323	0·0388	1·95	0·0188	1·8825	0·4500
Aluminium	132	0·098	0·0656	1·0	0·0318	1·8825	0·4500
Brass	64	0·308	0·1356	6·5	0·0658	1·8825	0·4502
Mild steel	31	0·283	0·2796	12·32	0·1358	1·8825	0·451
Stainless steel	9·4	0·283	0·9220	40·6	0·448	1·8825	0·456

properties than of thermal conductivity and density when deciding the most suitable material for a given application.

It appears from Table 1 that for given heat-transfer properties the fin length and fin effectiveness are independent of the material used if the fin has optimum dimensions. This is difficult to prove purely analytically, rather than by the analytical plus numerical methods used in obtaining Fig. 2 and Table 1, because of the complex nature of equation (4). However, from equation (5), if either fin length or fin effectiveness are independent of the fin material used then both are independent of the fin material. Making use of equations (2), (3) and (5) it is possible to re-arrange equation (4) into the following form:

$$2 - \frac{q_{tot}}{2\pi\theta_0 kb} + \frac{4\pi r_0^2 h\theta_0}{q_{tot}} - \frac{4\pi r_0^2 h\theta_0}{q_{tot}}\left[\frac{I_0(Z_0)\,K_1(Z_0) + K_0(Z_0)\,I_1(Z_0)}{I_0(Z_0)\,K_1(Z_1) + K_0(Z_0)\,I_1(Z_1)}\right]\left\{\frac{1 + (q_{tot}/\pi r_0^2 h\eta\theta_0)}{\sqrt{[1 + (q_{tot}/2\pi r_0^2 h\eta\theta_0)]}}\right\}\frac{\theta_1}{\theta_0} = 0 \qquad (6)$$

where

$$\frac{\theta_1}{\theta_0} = \frac{I_0(Z_1)\,K_1(Z_1) + K_0(Z_1)\,I_1(Z_1)}{I_0(Z_0)\,K_1(Z_1) + K_0(Z_0)\,I_1(Z_1)}$$

$$Z_0 = (2h/kb)^{\frac{1}{2}}\,r_0$$

and

$$Z_1 = [2h(1 + q_{tot}/2\pi r_0{}^2 h\eta\theta_0)/kb]^{\frac{1}{2}}\,r_0$$

There are only two terms in equation (6) which can be dependent upon the fin material used, they are the fin effectiveness η and the product kb, i.e. the product of fin thermal conductivity k and optimum fin thickness b. The products kb as obtained from Table 1, are found to be independent of the fin material used, but, of course, Table 1 is obtained by analytical plus numerical methods. However, it can be argued from equation (2) that kb and η are independent of fin material, because if they are dependent then

$$\frac{I_1(Z_0)\,K_1(Z_1) - K_1(Z_0)\,I_1(Z_1)}{I_0(Z_0)\,K_1(Z_1) + K_0(Z_0)\,I_1(Z_1)} = \frac{\text{function }(q_{tot}, h, r_0, \theta_0)}{\sqrt{(kb)}} \qquad (7)$$

From the very nature of modified Bessel functions it is impossible for equation (7) to be correct, modified Bessel functions are as follows:

$$I_0(Z) = \sum_{m=0}^{r} \frac{(\frac{1}{2}Z)^{2m}}{(m!)^2}$$

$$I_1(Z) = \sum_{m=0}^{\infty} \frac{(\frac{1}{2}Z)^{2m+1}}{m!\,m+1!}$$

$$K_0(Z) = -\,[\log(\tfrac{1}{2}Z) + \gamma]\,I_0(Z) + \sum_{r=0}^{\infty}\frac{(\frac{1}{2}Z)^{2r}}{(r!)^2}\left(\sum_{m=1}^{r} m^{-1}\right)$$

$$K_1(Z) = +\,(\log(\tfrac{1}{2}Z) + \gamma)\,I_1(Z) - \tfrac{1}{2}\sum_{r=0}^{\infty}\frac{(\frac{1}{2}Z)^{2r+1}}{r!\,r+1!}\left(\sum_{m=1}^{r+1} m^{-1} + \sum_{m=1}^{r} m^{-1}\right) + \frac{1}{Z}$$

where γ is Euler's number. Thus the only conclusion that can be drawn from equation (2) is that both the product kb and the fin effectiveness η are independent of the fin material used if the fin has optimum dimensions. From equation (5) the optimum fin length is also independent of the fin material used. A further argument in favour of this independence of the fin material can be obtained by considering the two limiting cases of equation (6), i.e. as hr_0^2 approaches zero or infinity. In both cases equation (6) reduces to

$$2 - \frac{q_{tot}}{2\pi\theta_0 kb} = 0$$

which clearly shows that kb is independent of fin material. It is of interest to note that the same independence of fin material is true of uniform and triangular straight fins, [1], [2], [3], and also of the annular fin of least material. The small variations in tip radius and fin effectiveness for the uniform annular fins of optimum dimensions are accounted for in the inaccuracies involved in interpreting Fig. 1. There is an overall scatter of about 1·2 per cent in tip radius and fin effectiveness.

The dimensions of annular fins of least material can be found from the two equations given in the introduction and from the following equation

$$V = \frac{\pi h r_0^4}{3k}\left(\frac{r_1}{r_0}+1\right)\left(\frac{r_1}{r_0}-1\right)^3$$

Applying these equations to the example used to illustrate the optimum dimensions of uniform annular fins, it is found that

$$r/_1r_0 = 1{\cdot}8825$$

$$b_0 = 4{\cdot}204/k \text{ in}$$

$$V = 8{\cdot}66/k \text{ in}^3$$

and

$$\eta = 7{\cdot}2/\text{surface area}$$

where the units of k are chu/h ft degC, b_0 is the fin bore thickness, the units of surface area are in^2, and where the surface area is given by

$$\text{surface area} = 4\pi\int_{r_0}^{r_1} r\sqrt{\left\{1+\frac{h^2r^2}{36k^2}\left[4-3\left(\frac{r_1}{r}\right)-\left(\frac{r_1}{r}\right)^3\right]^2\right\}}\,\mathrm{d}r$$

Substituting the appropriate values or r_1, r_0 and h into the expression for surface area it is found that

$$\eta = 7{\cdot}2/(15{\cdot}99 - 17{\cdot}65/k^2)$$

where the units of k are chu/h ft degC. The dimensions of the fins of least material are listed in Table 2 and a comparison between the dimensions of the annular fin of least material and the optimum dimensions of the uniform annular fin is given in Table 3. The uniform annular fin has a much greater volume but is considerably shorter than the annular fin of least material and is thinner at the bore.

If the ability to dissipate heat is the only criterion used to measure annular fin performance then the fin of least material is obviously the best. However, many other engineering aspects, not least of which are manufacturing processes, must be considered before a choice of fin shape and material can be made. It is felt that from a practical point of view, the uniform annular fin is the most superior and it is essential that the optimum dimensions of such fins can be determined.

The assumption that the heat flux from the tip of the fin is zero, see equation (1), is an approximation which makes the optimisation process basically invalid. The approximation underestimates the total amount of heat which can be dissipated by the fin and overestimates the fin temperatures. The ratio of the heat lost from the fin tip q_1 to the total heat dissipated from the fin surfaces q_{tot} gives an indication of the validity of the optimization process where

$$q_1 = 2\pi r_1 b h \theta_1$$

Table 3. A comparison between the optimum dimensions of uniform annular fins and the dimensions of the annular fin of least material

Material	Volume of uniform fin / Volume of least material fin	Thickness of uniform fin / Bore thickness of least material fin	Tip radius of uniform fin / Tip radius of least material fin
Copper	1·76	0·638	0·898
Aluminium	1·8	0·629	0·909
Brass	1·65	0·563	0·909
Mild steel	1·64	0·566	0·904
Stainless steel	1·59	0·501	0·904

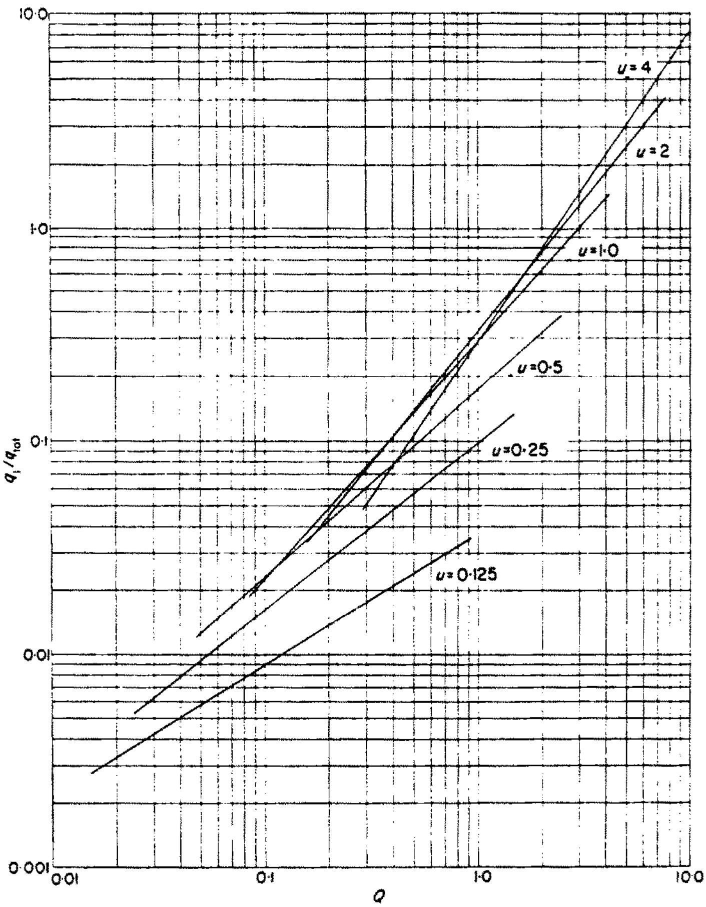

FIG. 3. Heat loss from the fin tip as a fraction of the heat dissipated from the fin surfaces.

Thus

$$\frac{q_1}{q_{tot}} = \frac{2\pi r_1 b h \theta_1}{q_{tot}} = \frac{\theta_1}{2\theta_0}\frac{u^2\sqrt{(1+uv)}}{wQ}$$

The ratio q_1/q_{tot} is plotted against Q for various values of u in Fig. 3. If, for example, one decides that a value of q_1/q_{tot} greater than 0·1 makes the optimization process invalid, then only the annular fin made of stainless steel listed in Table 1 would be in error.

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Résumé—Uné équation est obtenue reliant les dimensions optimales d'ailettes annulaires uniformes au transport de chaleur et aux propriétés thermiques de l'ailette et au coefficient de transport de chaleur entre l'ailette et le réfrigérant. Quelques résultats calculés couvrant convenablement de larges gammes de propriétés de l'ailette et le coefficient de transport de chaleur vers le réfrigérant sont présentés graphiquement en fonction de paramètres sans dimensions convenables. Le choix des paramètres rend possible la détermination des dimensions optimales d'une ailette, c'est-à-dire la largeur, la longueur et le volume, en fonction de la chaleur totale à dissiper par l'ailette, de sa conductivité thermique et du rayon du trou, du coefficient de transport de chaleur entre l'ailette et le réfrigérant, et de la différence de température entre le trou d'ailette et le réfrigérant. Les résultats généralisés sont appliqués à un exemple spécifique qui montre les effets des propriétés du matériau sur les dimensions optimales. Finalement, une comparaison est faite entre les dimensions optimales d'ailettes uniformes et les dimensions d'ailettes contenant le moins de matériau.

Zusammenfassung—Es wird eine Gleichung abgeleitet, welche die optimalen Abmessungen einer Ringrippe gleichbleibender Breite zum Wärmeübergang und den thermischen Eigenschaften der Rippe und der Wärmeübergangszahl zwischen Rippe und Kühlmittel in Beziehung setzt. Einige Berechnungsergebnisse, die einen ziemlich weiten Bereich der Eigenschaften der Rippe und der Wärmeübergangszahl des Kühlmittels überdecken, werden graphisch in Ausdrücken geeigneter, dimensionsloser Parameter angegeben. Die Auswahl von Parametern ermöglicht eine optimale Rippendimensionierung, d.h. Breite, Länge und Volumen, in Termen der von der Rippe abzuleitenden Gesamtwärme, ihres Wärmeleitvermögens und ihres Innenradius, der Wärmeübergangszahl zwischen Rippe und Kühlmittel und der Temperaturdifferenz zwischen Rippenwurzel und Kühlmittel. Die verallgemeinerten Ergebnisse werden auf ein besonderes Beispiel angewandt, das den Einfluss der Stoffwerte auf die optimalen Abmessungen zeigt. Schliesslich wird die Optimalabmessung der Rippe mit gleichbleibender Breite mit der Abmessung einer Rippe mit geringstem Materialaufwand verglichen.

Аннотация—Выведено уравнение, связывающее оптимальные размеры однородных круглых ребер с процессом переноса тепла, теплофизическими свойствами ребер и коэффициентом переноса тепла между ребром и охладителем. Некоторые рассчитанные результаты, охватывающие довольно широкий диапазон свойств ребер и коэффициента теплообмена охладителя, представлены графически для некоторых безразмерных параметров. Выбор параметров позволяет определить оптимальные размеры ребра, т.е. ширину, длину и объём, через полное количество тепла, отдаваемое ребром, его теплопроводность, радиус отверстия, коэффициент теплообмена между ребром и охладителем и разность температур отверстия ребра и охладителя. Обобщенные результаты иллюстрировались на специальном примере, который показывает влияние свойств материала на оптимальные размеры. Наконец, сделано сравнение между оптимальными размерами однородных ребер и размерами ребер из наименьшего количества материала.